PMM U.S.S.R.,Vol.49,No.5,pp. 595-601,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

THE TWO-DIMENSIONAL INVERSE PROBLEM OF ELASTICITY THEORY FOR INHOMOGENEOUS MEDIA IN POLAR COORDINATES*

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The following problem is consiered: it is required to find the distribution law for the elasticity parameters of the material in a body in a given state of stress. Three modifications of this problem are derived: 1) the stresses and shear modulus are given, find the law of Poisson's ratio variation in the body; 2) the stresses and Poisson's ratio are given, find the nature of the change in the shear modulus of the body material; 3) find the set of functions describing the law of variation in two parameters of the material elasticity for a known state of body stress.

It is shown that the first kind of inverse problem reduces to solving Poisson's equation. The other two result in a second-order partial differential equation with variable coefficients. Its solutions are investigated for states of stress with zero shear stresses. As is known from classical elasticity theory, states of stress of this kind can occur in bodies in the form of a long pipe subjected to internal and external pressure, in the pure bending of a circular bar, in a wedge, in problems of stress concentration around holes, etc.

An extensive class of particular solutions is obtained for the second kind of inverse problem when no constraints, with the exception of integrability of the given functions, can be imposed on the stress.

It is shown that in solving the third kind of inverse problems, when both elasticity parameters are variable, the laws of their variation can be expressed in terms of one arbitrary function.

Such problems were first investigated in /1, 2/. Solutions obtained up to now refer mainly to bodies of the simplest shapes and elementary states of stress. Thus, rectangular elements were considered in /1/ with states of stress of the pure shear, tension-compression, or bending types. The problem of seeking the law of Young's modulus variation in a wedge with a radial stress distribution. An approximate solution is obtained in /4/ for the two-dimensional problem of a long cylinder. Surveys of the researches devoted to the problem under consideration are given in bibliographies /5, 6/ and monographs /7, 8/.

The whole range of problems associated with seeking the law of elasticity parameter variation for a material according to given stresses is customarily called the "inverse problem" in the theory of the elasticity of inhomogeneous media. Such a definition cannot encompass all possible formulations of inverse problems, and it must be supplemented. To do this, by analogy with classical elasticity theory, we separate the whole manifold of problems into three groups by isolating first, second, and mixed inverse problems of the mechanics of inhomogeneous media. The final purpose in solving each of them is to seek the elasticity parameter distribution law in a body and the problems are distinguished just by the initial data. In the first inverse problem, the stresses in the body are considered given, while in the second it is the displacements, and in the mixed problem the separate stress tensor components and the displacement vector are prescribed.

The distinctions in the initial data also predetermine the substantial differences in the procedure for the solutions. The first inverse problem of the theory of elasticity of inhomogeneous bodies is examined below.

1. To solve the first inverse problem of the theory of elasticity of inhomogeneous media it is necessary to solve the continuity equation in which the strains are expressed in terms of the stress. In a polar coordinate system $r = \sqrt{x^2 + y^2}$, $\beta = \operatorname{arc\,tg}(y/x)$ this equation can be written in the form

$$\left[\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial\beta^2}\right]\left[\frac{1-2\nu}{2G^*}(\sigma_r + \sigma_\beta)\right] - \left[\left(r\frac{\partial}{\partial r}\right)^2 + 2r\frac{\partial}{\partial r} - \frac{\partial^2}{\partial\beta^2}\right]\frac{\sigma_r - \sigma_\beta}{2G^*} - 2\frac{\partial}{\partial\beta}\left(r\frac{\partial}{\partial r} + 1\right)\frac{\tau_{r\beta}}{G^*} = 0 \tag{1.1}$$

$$G^* = \frac{G}{G_0}$$

*Prikl.Matem.Mekhan.,49,5,775-783,1985

Here $\sigma_r, \sigma_\beta, \tau_{r\beta}$ are the stress tensor components, $\nu = \nu(r, \beta)$ is Poisson's ratio, $G^* = G^*(r, \beta)$ is the relative shear modulus, and G_0 is the value of the shear modulus at some point of the body. Eq.(1.1) has is valid for the case of plane strain. For the plane state of stress, ν must be replaced by $\nu/(1+\nu)$.

Analysis of (1.1) shows that three modifications are possible for the first inverse problem: 1) $\sigma_r, \sigma_\beta, \tau_{r\beta}$ and $G(r, \beta)$ are given, find the law of variation of Poisson's ratio $\nu(r, \beta)$ in the domain S occupied by the body; 2) the stresses and the function $\nu(r, \beta)$ are given, find the law of variation of the shear modulus $G(r, \beta)$; 3) only the stresses are known and the set of functions $G(r, \beta)$ and $\nu(r, \beta)$ must be found for which a given state of stress will be realized.

2. The first case is the simplest. To solve this kind of inverse problem, (1.1) should be written in the form (Δ is the two-dimensional Laplace operator)

$$\Delta N = Q(r, \beta), \quad N = \frac{1-2\nu}{G^*}(\sigma_r + \sigma_\beta) \tag{2.1}$$

$$Q = \left[\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \beta^2}\right]\frac{\sigma_r - \sigma_\beta}{G^*} + 4\frac{\partial}{\partial \beta}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\right)\frac{\tau_{r\beta}}{G^*}$$

By having the functions $\sigma_r, \sigma_\beta, \tau_{r\beta}$ and $G^*(r, \beta)$ available, Q can be determined and the problem of seeking $\nu(r, \beta)$ reduces to solving Poisson Eq.(2.1). Let $\psi(r, \beta)$ denote some particular solution of (2.1), and $\Psi(r, \beta)$ an arbitrary harmonic function. Then from (2.1) it follows that

$$\nu(r, \beta) = \frac{1}{2} - \frac{G^*(r, \beta)}{2(\sigma_r + \sigma_\beta)}(\Psi + \psi)$$

3. We now assume that some particular solution of the inverse problem has been found successfully. We let $G^*(r, \beta)$ and $\nu^0(r, \beta)$ denote functions describing the nature of the change in parameters of the material elasticity in the domain S occuplied by the body. We will show that by having such a solution available we can transfer to a more general case of inhomogeneity.

Let $\nu^+(r, \beta)$ be the law of variation of Poisson's ratio not taken into account by the function $\nu^\circ(r, \beta)$. Here the same stresses as for Poisson's ratio $\nu^\circ(r, \beta)$ occur in the body with relative shear modulus $G^*(r, \beta)$ Then the more general case of the inhomogeneities will be described by the functions

$$G^* = G^*(r, \beta), \quad \nu = \nu^\circ(r, \beta) - \nu^+(r, \beta) \tag{3.1}$$

The problem therefore is to seek $\nu^+(r, \beta)$.

Such a formulation of the problem is similar in form to that considered in Sect.2. Substituting the function (3.1) and the given stresses $\sigma_r, \sigma_\beta, \tau_{r\beta}$ into (2.1) and taking into account that G^* and ν° are its particular solutions, we obtain $\Delta\,[\nu^+(\sigma_r + \sigma_\beta)/G^*] = 0$. Hence it follows that

$$\nu^+ = G^*(\sigma_r + \sigma_\beta)^{-1}\,\Psi(r, \beta) \tag{3.2}$$

4. The solution of problems corresponding to other modifications of the first inverse problem is fraught with mathematical difficulties. To simplify the problem, we will carry out further investigations as they apply to the special case of the state of stress when $\tau_{r\beta} \equiv 0$ in the domain S occupied by the body.

We will first examine the case when only the stresses are known and the set of functions $G^*(r, \beta), \nu(r, \beta)$ must be found for which a given state of stress is realized.

We introduce the function $\Phi = \Phi(r, \beta)$ into consideration such that

$$G^* = \frac{\sigma_r - \sigma_\beta}{\Delta_1\Phi}, \quad \nu = \frac{1}{2} - \frac{(\sigma_r - \sigma_\beta)\,\Delta_2\Phi}{2(\sigma_r + \sigma_\beta)\,\Delta_1\Phi} \tag{4.1}$$

$$\left(\Delta_1 = \left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial \beta^2}, \quad \Delta_2 = \left(r\frac{\partial}{\partial r} + 1\right)^2 - \frac{\partial^2}{\partial \beta^2} - 1\right)$$

Substituting these relations into (1.1), we see that for $\tau_{r\beta} \equiv 0$ it is satisfied identically for any selection of the function Φ. Therefore, relations (4.1) yield a general solution of the inverse-problem type of the theory of elasticity of inhomogeneous media under consideration. Only those functions G^* and ν which satisfy the conditions

$$0 < G < \infty, \quad 0 \leqslant \nu \leqslant {}^1/_2 \tag{4.2}$$

in the domain occupied by the body, evidently have any physical meaning.

Solving the equilibrium equation for $\tau_{r\beta} \equiv 0$, we find that stresses that can be represented in the form

$$\sigma_\beta = \sigma(r), \quad \sigma_r = \frac{\varphi_1(\beta)}{r} + \frac{1}{r}\int\limits_{\varphi_2(\beta)}^{r} \sigma(r)\,dr \tag{4.3}$$

are allowable, where $\sigma(r)$, $\varphi_{1,2}(\beta)$ are arbitrary functions.

5. The inverse problem in which the law of variation of the shear modulus $G^*(r, \beta)$ must be found for known stresses and Poisson's ratio $\nu(r, \beta)$ is the most difficult one since a partial differential equation with variable coefficients (1.1) must be solved, which cannot, as a rule, be successfully reduced to those studied.

We investigate this equation for $\tau_{r\beta} \equiv 0$. Moreover, we first assume that Poisson's ratio and the stresses are known functions of one coordinate r.

We write (1.1) in the form (the prime denotes differentiation with respect to r):

$$F'' + \frac{f+1}{r} F' + \frac{f'}{r} F - \frac{f-1}{r^2} \frac{\partial^2 F}{\partial \beta^2} = 0$$

$$\left(f = \frac{\sigma_r - \sigma_\beta}{\nu\sigma_r - (1-\nu)\sigma_\beta}, \quad F = \frac{\nu\sigma_r - (1-\nu)\sigma_\beta}{G^*}\right)$$

Using the method of separation of variables $F = R_m(r) B_m(\beta)$, we obtain two ordinary linear differential equations to seek the functions $R_m(r)$ and $B_m(\beta)$ (m is a numerical parameter)

$$\frac{d^2 B_m}{d\beta^2} + m^2 B_m = 0 \tag{5.1}$$

$$R_m'' + \frac{f+1}{r} R_m' + \frac{1}{r}\left(f' + m^2 \frac{f-1}{r}\right) R_m = 0 \tag{5.2}$$

Integrating (5.1), we have (A_{1m}, A_{2m} are arbitrary constants)

$$B_m = A_{1m} \cos m\beta + A_{2m} \sin m\beta, \quad m \neq 0; \quad B_0 = A_{10} + A_{20}\beta \tag{5.3}$$

We now consider (5.2). The general integral of this equation can be indicated /9/ for many of the simplest dependences $f(r)$, however, the problem of seeking the solution at least for the individual values of the parameter m and Poisson's ratio when no other constraints with the exception of integrability need be imposed on the function $f(r)$ is of greatest interest, and would enable us to investigate a number of inverse problems for bodies with arbitrary stress fields of the form (4.3). Even the case when the stresses $\sigma_\beta = \sigma(r)$ change by jumps form one finite value to another for certain r could be examined.

Such particular values for the parameter m would be $m = 0$ and $m = 1$, where no constraints, with the exception of (4.2), are imposed on $\nu(r)$

Indeed, in this case the equation for R_m can be written in the form

$$\left(\frac{d}{dr} + \frac{m+1}{r}\right)\left(\frac{d}{dr} + \frac{f-m}{r}\right) R_m = 0 \quad (m = 0, 1) \tag{5.4}$$

i.e., it decomposes into two first-order equations which, when solved, yeild (C_{1m} and C_{2m} are arbitrary constants)

$$R_m = r^m e^{-\varphi}\left(C_{1m} + C_{2m} \int r^{-(2m+1)} e^{\varphi}\, dr\right), \quad \varphi = \int f \frac{dr}{r} \tag{5.5}$$

We will present still another mode of writing the solution of (5.4) in the case when the stresses and Poisson's ratio in the body are such that

$$\frac{f-m}{r} = \frac{Q_1(r)}{Q_2(r)} \quad (m = 0, 1) \tag{5.6}$$

where $Q_{1,2}(r)$ are polynomials, the degree of the polynomial $Q_2(r)$ is greater than the degree of $Q_1(r)$, and the right side of (5.6) is an irreducible fraction which, it is known, can be converted into a sum of elementary fractions

$$\frac{f-m}{r} = \frac{a_1}{r-r_1} + \frac{a_2}{r-r_2} + \ldots + \frac{a_n}{r-r_n} \quad (Q_2(r_i) = 0, \ i = 1, 2, \ldots)$$

The solution of (5.4) can be represented in a form equivalent to the solution (5.5)

$$R_m = (r-r_1)^{-a_1}(r-r_2)^{-a_2} \ldots (r-r_n)^{-a_n}\left[C_{1m} + C_{2m} \int r^{-(m+1)} (r-r_1)^{a_1} \ldots (r-r_n)^{a_n}\, dr\right] \tag{5.7}$$

Thus, the general integral of (5.2) has the form (5.5) for $m = 0, 1$ and arbitrary $\sigma_\beta(r)$, $\sigma_r(r)$, $\nu(r)$, and in the special case when the function $(f - m)/r$ is of the type (5.6) can be represented in the form (5.7).

The law of shear modulus variation in a body has the form

$$\frac{1}{G^*} = \frac{1}{\nu\sigma_r - (1-\nu)\sigma_\beta} \sum_{m=0}^{1} R_m(r) B_m(\beta) \tag{5.8}$$

Having this solution available, a more general case of inhomogeneity can be found by using relationships (3.1) and (3.2).

If the body material is incompressible $(\nu = 1/2)$, and operates under plane strain conditions, then $f = 2$ for any σ_r and σ_β given in the form (4.3). Equation (5.2) is converted to the form

$$R_m'' + \frac{3}{r} R_m' + \frac{m^2}{r^2} R_m = 0$$

Its solution is

$$\begin{gathered} R_m = r^{-1} (C_{1m} \cos n\rho + C_{2m} \sin n\rho), \quad n = \sqrt{m^2 - 1} \\ \rho = \ln r, \quad m \neq 0, 1 \\ R_0 = C_{10} + C_{20} r^{-2}, \quad R_1 = r^{-1} (C_{11} + C_{21} \ln r) \end{gathered} \tag{5.9}$$

Therefore, the law of shear modulus variation in the case of the plane strain of an inhomogeneous body of incompressible material can formally be represented in the form of the series

$$\frac{1}{G^*} = \frac{1}{\sigma_r - \sigma_\beta} \sum_{m=0}^{\infty} R_m(r) B_m(\beta) \tag{5.10}$$

By using the relationships (3.1) and (3.2) it is possible to pass from this solution to the more general case of an inhomogeneity when the shear modulus in the body is described by the dependence (5.10) and Poisson's ratio

$$\nu = \frac{1}{2} - \frac{G^*}{\sigma_r + \sigma_\beta} \Psi(r, \beta) \tag{5.11}$$

Formula (5.11) is obtained for the case of plane strain. For the generalized plane state of stress the ν must be replaced by $\nu/(1 + \nu)$. Finally, the law of variation of Poisson's ratio in a body takes the form

$$\nu = \frac{\sigma_r + \sigma_\beta - 2G^*\Psi(r, \beta)}{\sigma_r + \sigma_\beta + 2G^*\Psi(r, \beta)} \tag{5.12}$$

6. If the relationships between the displacements, strains, and stresses known in the theory of elasticity are used, then the components of the displacement vector u_r and u_β can be determined.

Thus, if new constants and notation are introduced

$$\chi_n^{\pm}(r) = \int e^{\pm\varphi} \frac{dr}{r^n}, \qquad U_{kl} = A_{k1} \cos\beta + A_{l1} \sin\beta$$
$$V_{kl} = A_{k1} \sin\beta - A_{l1} \cos\beta$$

then from relations (5.3), (5.5) and (5.8), the law of shear modulus variation for $\nu = 1/2$ and $m = 0$ can be written in the form

$$\frac{1}{G^*} = \frac{e^{-\varphi}}{(1-\nu)\sigma_\beta - \nu\sigma_r} [A_{10} + A_{20}\beta + (A_{30} + A_{40}\beta)\chi_1^+(r)] \tag{6.1}$$

We hence find

$$\begin{gathered} 2G_0 u_r = (A_{10} - A_{20}\beta)\, r e^{-\varphi} + (A_{30} + A_{40}\beta)\, r\, [e^{-\varphi}\chi_1^+(r) - 1] + \\ a \sin\beta - b\cos\beta \\ 2G_0 u_\beta = -A_{20} r \chi_1^-(r) - A_{40} r \left[\int e^{-\varphi}\chi_1^+(r) \frac{dr}{r} - \ln r\right] + \\ \left(A_{30}\beta - A_{40}\frac{\beta^2}{2}\right) r + cr + a\cos\beta - b\sin\beta \end{gathered} \tag{6.2}$$

For $m = 1$ we have

$$\frac{1}{G^*} = \frac{re^{-\varphi}}{\nu\sigma_r - (1-\nu)\sigma_\beta} [U_{12}(\beta) + \chi_3^+(r) U_{34}(\beta)] \tag{6.3}$$

$$\begin{gathered} 2G_0 u_r = [\chi_{-1}^-(r) - r^2 e^{-\varphi}]\, U_{12}(\beta) + \\ \left[\int e^{-\varphi}\chi_3^+(r)\, r\, dr - r^2 e^{-\varphi}\chi_3^+(r) + \ln r - \frac{1}{2}\right] U_{34}(\beta) + \frac{\beta}{2} V_{34}(\beta) \end{gathered} \tag{6.4}$$

$$2G_0 u_\beta = -\chi_{-1}^-(r) V_{12}(\beta) - \left[\int e^{-\varphi}\chi_3^+(r)\, r\, dr + \ln r\right] V_{34}(\beta) + 1/2\beta U_{34}(\beta)$$

Analogous relationships can be obtained for $\nu = 1/2$ and any m.

7. We examine the first inverse problem for a long inhomogeneous cylinder under plane strain conditions subjected to an internal pressure p_0 and an external pressure p_1 (a and b are the cylinder inner and outer radii). We limit ourselves to the case when the function $G^*(r, \beta)$ must be sought for a known Poisson's ratio.

If the cylinder material is homogeneous, then, as Lamé showed, the state of stress is characterized by the absence of shear stresses. We assume that $\tau_{r\beta} \equiv 0$ also in the case of an inhomogeneous material while the remaining stress tensor components (σ_r, σ_β) depend only on r.

Moreover, we a-sume that $\nu \neq {}^1/_2$.

As has been shown above, (5.2) for an arbitrary function $\nu(r)$ is integrated successfully only for two values of the parameter m, where the shear modulus can be represented in the form (5.5) or, equivalently, in the form of a sum of the functions (6.1) and (6.3).

Arbitrary constants occur in the solution of the inverse problem, and to avoid uncertainty, it must be supplemented by appropriate boundary conditions.

It follows from the formulas for the displacements (6.2) and (6.4) that the displacements will be determined uniquely in the case under consideration if we set

$$A_{20} = A_{30} = A_{40} = A_{31} = A_{41} = 0 \tag{7.1}$$

The remaining arbitrary constants can be determined by requiring that the function $G^*(r, \beta)$ takes given values on one of the domain boundaries, for $r = a$, say.

It follows from relationships (6.1) and (6.3) that the inverse problem can be solved only for the following boundary conditions ($\alpha_0, \alpha_1, \alpha_2$ are constants):

$$r = a, \quad 1/G^* = \alpha_0 + \alpha_1 \cos\beta + \alpha_2 \sin\beta$$

An an illustration, consider the inverse problem for an inhomogeneous cylinder with $\nu = \text{const}$ and the boundary condition

$$r = a, \quad G^* = 1 \tag{7.2}$$

The condition is independent of β and consequently (6.1) and (6.2), which correspond to the case $m = 0$, can be bypassed.

In engineering analyses of pipes under internal and external pressure, it is often assumed that $\sigma_\beta = c_1$ to simplify the computations. We will investigate the possible nature of cylinder inhomogeneities under this assumption.

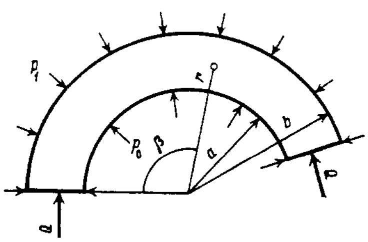

Fig.1

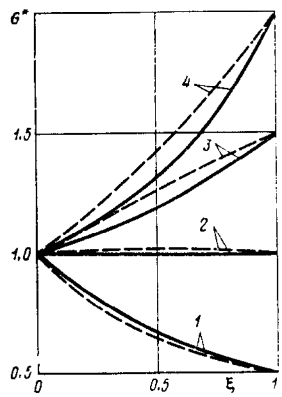

Fig.2

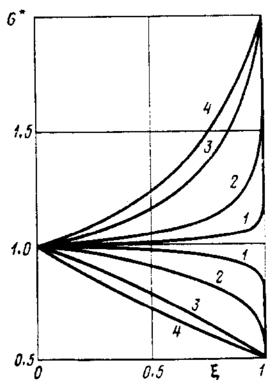

Fig.3

It follows from (4.3) and the boundary conditions that

$$\sigma_r = \frac{c_0}{r} + c_1, \quad c_0 = \frac{p_1 - p_0}{b - a}\, ab, \quad c_1 = \frac{p_0 a - p_1 b}{b - a} \tag{7.3}$$

For $\nu \neq 0,\ 1/2$ we have

$$\frac{f}{r} = \frac{c_0}{\nu c_0 r - (1 - 2\nu) c_1 r^2} = \nu^{-1}\left[r^{-1} + \left(r - \frac{\nu}{1 - 2\nu}\alpha\right)^{-1}\right] \quad \left(\alpha = \frac{c_0}{c_1}\right) \tag{7.4}$$

Taking account of (6.1), (7.1) and (7.2), we have from (5.7) and (5.8)

$$\frac{1}{G^*} = A_{10}\left[1 - \frac{\nu\alpha}{(1 - 2\nu) r}\right]^{1/\nu - 1} \quad \left(A_{10} = \left[1 - \frac{\nu\alpha}{(1 - 2\nu) a}\right]^{1 - 1/\nu}\right)$$

Values of G^* are presented below which show the nature of the change in the relative shear modulus over the pipe thickness for different $\nu = 0.1, 0.3, 0.4$ for $p_1 = 0$ and $b : a = 3 : 2$ ($\xi = (r - a)/(b - a)$, $G^* = 1$ for $\xi = 0$)

ν	$\xi = 0{,}2$	0,4	0,6	0,8	1
0,1	1,096	1,183	1,263	1,337	1,404
0,3	1,097	1,189	1,275	1,356	1,433
0,4	1,098	1,193	1,285	1,373	1,458

It is seen that the change in ν has a neglibile effect on the function $G^*(r)$.

8. If the inverse problem is sovled for a simply-connected domain, then the requirement (7.1) cannot be imposed. A number of important effects of the influence of the nature of the inhomogeneities on the state of stress of the body can here be clarified successfully.

We will solve the inverse problem for an inhomogeneous curved bar occupying the domain S: $[a < r < b,\ 0 < \beta < \alpha^*]$. Compressive forces p_0 and p_1 act on the inner and outer faces of the bar while normal forces with resultant Q (Fig.1) act on the endfaces $\beta = 0$ and $\beta = \alpha^*$. We will confine ourselves to the case when the function $G^*(r, \beta)$ must be sought for a known Poisson's ratio.

It follows from the dependences (6.1) and (6.3) that for $\nu \neq 1/2$ the inverse boundary value problem can be solved for the following boundary conditions on the faces:

$$r = a,\quad 1/G^* = \alpha_0 + \alpha_1 \cos\beta + \alpha_2 \sin\beta + \alpha_3\beta$$
$$r = b,\quad 1/G^* = \gamma_0 + \gamma_1 \cos\beta + \gamma_2 \sin\beta + \gamma_3\beta \tag{8.1}$$

If Poisson's ratio is $\nu = 1/2$ or varies in conformity with the dependence (5.11) within the limits of the body, then the problem under consideration is solved successfully for the most general case of the boundary conditions

$$r = a,\quad G^* = G_0^*(\beta);\quad r = b,\quad G^* = G_1^*(\beta) \tag{8.2}$$

Indeed, in this case the shear modulus can be written in the mode (5.10).

Expressing $B_m(\beta)$ in terms of trigonometric functions for $A_{1m} = 1, A_{2m} = 0$, we will substitute the dependence (5.10) into the boundary conditions (8.2). We consequently find

$$\left\| \begin{matrix} 1/G_0^* \\ 1/G_1^* \end{matrix} \right\| = \sum_{k=0}^{\infty} \cos m\beta \left\| \begin{matrix} H_m(a) \\ H_m(b) \end{matrix} \right\| \left(H_m(r) = \frac{R_m(r)}{\sigma_r - \sigma_\beta},\ m = \frac{2k\pi}{\alpha^*} \right) \tag{8.3}$$

Now expanding the left side in a Fourier series

$$\left\| \begin{matrix} 1/G_0^* \\ 1/G_1^* \end{matrix} \right\| = \sum_{k=0}^{\infty} \cos m\beta \left\| \begin{matrix} \alpha_{0k} \\ \alpha_{1k} \end{matrix} \right\|$$

and collecting like terms for all $\cos m\beta$, we equate them to zero. For each k we consequently obtain simple algebraic equations to determine all the arbitrary constants that enter into the function $H_m(r)$.

As an illustration, we consider the first inverse problem for a curved bar with $\nu = \text{const}$ and the following boundary conditions:

$$r = a,\ G^* = 1;\quad r = b,\ G^* = G_1^* \quad (G_1^* = \text{const}) \tag{8.4}$$

The conditions are independent of β and, consequently, the law of shear modulus variation can be represented in the form (6.1) for $A_{20} = A_{40} = 0$.

We will investigate the case when the state of stress of an inhomogeneous bar agrees with the state of stress of a homogeneous bar, i.e., has the form

$$\sigma_r = \frac{A}{r^2} + B,\quad \sigma_\beta = -\frac{A}{r^2} + B,\quad A = \frac{a^2b^2(p_1 - p_0)}{b^2 - a^2},\quad B = \frac{a^2p_0 - b^2p_1}{b^2 - a^2} \tag{8.5}$$

Then

$$\frac{f}{r} = \frac{2A}{r[A - (1-2\nu)Br^2]} = \frac{2}{r} - \frac{1}{r - r_0} - \frac{1}{r + r_0}$$
$$r_0 = \left[\frac{A}{(1-2\nu)B}\right]^{1/2}$$

Therefore, the function f/r is of the type (5.6), which enables (5.7) to be used. Having determined the arbitrary constants from (8.4), we finally find

$$G^* = \left[1 + \frac{1 - G_1^*}{G_1^*}\frac{T(r, a)}{T(b, a)}\right]^{-1} \left(T(r, a) = \ln\frac{A - (1-2\nu)Br^2}{A - (1-2\nu)Ba^2}\right) \tag{8.6}$$

We show by the solid lines in Fig.2 the nature of the change in the shear modulus G^* for $\nu = 1/3$, $b : a = 3 : 2$ and different $G_1^* = 0.5, 1, 1.5, 2$ (curves *1–4* respectively). The bar is subjected to the pressure p_1 while $p_0 = 0$.

It follows from (8.6) that the solution of the problem under consideration with the boundary conditions (8.4) always exists, with the exception of the case when the function $A - (1-2\nu)Br^2$ changes sign by passing through zero in the domain S. However, a trivial solution $G^* = \text{const}$ of (1.1) is always possible here.

We investigate the nature of the change in the shear modulus in a body when the quantity $A - (1-2\nu)Br^2$ vanishes on the boundary of the domain $S: [a < r < b,\ 0 < \beta < \alpha^*]$ for $r = l$ ($b = l - \varepsilon$, where ε is a small quantity). This is possible if $\nu \neq 1/2$ and

$$\frac{p_1}{p_0} = \left[1 + (1-2\nu)\frac{l^2}{b^2}\right]\left[1 + (1-2\nu)\frac{l^2}{a^2}\right]^{-1}$$

In the case under consideration $T(r, a) = \ln[(l^2 - r^2)/(l^2 - a^2)]$.

We show in Fig.3 the nature of the change in the shear modulus in a body for $\nu = 1/3$, $\varepsilon =$ [illegible] 10^{-3}, 10^{-4}, [illegible] curves *1–*[illegible] respectively for the case $b : a = 8 : 2$ and two values of the relative shear modulus at the pipe outer surface: $G_1^* = 0.5$ (curves located below the line $G^* = 1$) and $G_1^* = 2$ (remaining curves). It is seen that the shear modulus starts to vary only as ξ approaches one, assuming the given value G_1^* for $\xi = 1$

It should be emphasized that all these inhomogeneity effects have no influence on the state of stress in the bar; it agrees with the state of stress in a homogeneous bar.

There are no such effects for $\nu = 1/2$ since the law of shear modulus variation takes the form

$$G^* = \left[1 + \frac{1 - G_1^*}{G_1^*} \frac{r^2 - a^2}{b^2 - a^2}\right]^{-1}$$

We investigate the case now when $\sigma_\beta = c_1 = \text{const}$. Then the functions σ_r and f/r have the form (7.3) and (7.4). We solve the problem under the boundary conditions (8.4).

From (6.1) we have

$$\frac{1}{G^*} = \left(1 - \frac{\nu\alpha}{(1-2\nu)\, r}\right)^{1/\nu - 1} \left[A_{10} + A_{30} \int \left(r - \frac{\nu\alpha}{1-2\nu}\right)^{-1/\nu} r^{1/\nu - 1}\, dr\right]$$

$$\left(\alpha = \frac{c_0}{c_1}\right)$$

The integral is solved by the method of rationalization for any $\nu = m/n$, where m and n are natural numbers. Thus, for $\nu = 1/3$ we obtain

$$\frac{1}{G^*} = \frac{(r-\alpha)^2}{r^2} \left[A_{10} + A_{30}\left(\ln|r-\alpha| - \frac{2\alpha}{r-\alpha} - \frac{\alpha^2}{2(r-\alpha)^2}\right)\right]$$

The arbitrary constants are determined from the boundary conditions (8.4).

The dashed curves in Fig.2 illustrate the nature of the change in the relative shear modulus over the bar thickness. The bar dimensions, Poisson's ratio of the material, and the boundary conditions are the same as in the first illustration.

If curves *1* or *2* are compared, we see that the functional dependences $G^*(r, \beta)$ to which they correspond are quite close. But, meanwhile, the states of stress of the bodies differ radically. The deduction can hence be made that in certain cases an insignificant change in the body inhomogeneity can result in a large change in the stress field.

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Translated by M.D.F.